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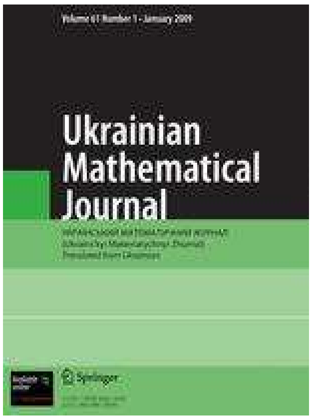

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ABSOLUTE SUMMATION OF SERIES BY THE ROGOSINSKI – BERNSTEIN METHOD

V. I. Kuz'mich

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1. A series $\sum_{n=0}^{\infty} u_n$ is said to be absolutely summable by the lower triangular matrix $A = (a_{nk})$ of a transformation of a series into a series (the A method) or $|A|$-summable to a number U if

$$\sum_{n=0}^{\infty}\left|\sum_{k=0}^{n} u_k a_{nk}\right| < \infty, \quad \sum_{n=0}^{\infty}\sum_{k=0}^{n} u_k a_{nk} = U.$$

The Rogosinski method is defined by the matrix $R = (r_{nk})$, where

$$r_{00} = 1, \ r_{nk} = \cos\frac{k\pi}{2n+2} - \cos\frac{k\pi}{2n} \ (0 \leqslant k \leqslant n).$$

The Rogosinski – Bernstein method is defined by the matrix $B = (b_{nk})$, where

$$b_{nk} = \cos\frac{k\pi}{2n+1} - \cos\frac{k\pi}{2n-1} \ (0 \leqslant k < n), \ b_{nn} = \cos\frac{n\pi}{2n+1} \ (n \geqslant 0).$$

The sequence of the numbers p_n $(n \geq 0)$, $p_0 \neq 0$, defines the Voron – Nörlund method with the matrix $W = (w_{nk})$, where $w_{nk} = P_{n-k}/P_n - P_{n-k-1}/P_{n-1}$ $(0 \leqslant k < n)$, $w_{nn} = P_0/P_n$ $(n \geqslant 0)$, $P_n = \sum_{k=0}^{n} p_k$ $(n \geqslant 0)$. In particular, for

$$w_{nk} = kA_{n-k}^{\alpha-1}/nA_n^{\alpha} \ (0 \leqslant k \leqslant n), \ A_m^{\beta} = (\beta+1)(\beta+2)\ldots(\beta+m)/m!$$

this is the Cesaro method of order $\alpha > -1$. We denote its matrix by C_α.

A method of summation is said to be absolutely regular if it absolutely sums each series that converges absolutely to a number U to the same number U. By the Knopp – Lorentz theorem [1, pp. 34, 35], the matrix method $A = (a_{nk})$ with the matrix of transformation of a series into a series is absolutely regular if and only if

$$\sum_{n=0}^{\infty} |a_{nk}| = O(1), \quad \sum_{n=0}^{\infty} a_{nk} = 1 \ (k \geqslant 0).$$

It is easily verified that the Rogosinski and the Rogosinski – Bernstein methods are absolutely regular.

Two methods are said to be absolutely equivalent if they absolutely sum the same series.

THEOREM 1. The Rogosinski – Bernstein method is absolutely equivalent to the Voron – Nörlund method defined by the sequence of numbers $p_n = 2$ $(n > 0)$, $p_0 = 1$.

Proof. The matrix $W = (w_{nk})$ of the Voron – Nörlund method has the form

$$w_{nk} = \frac{4k}{4n^2-1} \ (0 \leqslant k < n), \ w_{nn} = \frac{2n-1}{4n^2-1} \ (n \geqslant 0).$$

To prove the theorem, by virtue of Lemma 2 of [2], it is sufficient to show that the matrix $BW^{-1} = (a_{nk})$ is equivalent to absolute convergence. For this we find the elements $\overline{w}_{nk}$ of the matfix W^{-1} – the inverse matrix of W. We have [1, pp. 57, 103] $\overline{w}_{nk} = (-1)^{n-k} 4k (0 \leqslant k \leqslant n-1)$, $\overline{w}_{nn} = 2n+1$ $(n \geqslant 0)$. Now $a_{nk} = \sum_{i=k}^{n} b_{ni}\overline{w}_{ik} = (2k+1)(\cos k\alpha_n - \cos k\alpha_{n-1}) - 4k\left(\sum_{i=0}^{n-k-2}(-1)^i \cos(k+i+1)\alpha_n - \cos(k+i+1)\alpha_{n-1}\right) + (-1)^{n-k-1}\cos n\alpha_n$ $(0 \leqslant k < n)$, $a_{nn} = (2n+1)\cos n\alpha_n$ $(n \geqslant 0)$, where $\sum_{i=0}^{-1} c_i = 0$ and $\alpha_n = \frac{\pi}{2n+1}$ $(n \geqslant 0)$.

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Using the result of [3, 1.341.6], after simplifications, we get

$$a_{nk} = \cos k\alpha_n + 2k \sin k\alpha_n \operatorname{tg}\frac{\alpha_n}{2} - \left(\cos k\alpha_{n-1} + 2k\sin k\alpha_{n-1}\operatorname{tg}\frac{\alpha_{n-1}}{2}\right)$$
$$(0 \leqslant k < n-1)$$

Let us set

$$\varphi_k(x) = \cos\frac{k\pi}{2x+1} + 2k\sin\frac{k\pi}{2x+1}\operatorname{tg}\frac{\pi}{2(2x+1)}.$$

We have

$$\varphi_k'(x) = \frac{2k\pi}{(2x+1)^2}\left(\left(1 - k\sec^2\frac{k\pi}{2(2x+1)}\right)\sin\frac{k\pi}{2x+1} - 2x\cos\frac{k\pi}{2x+1}\operatorname{tg}\frac{\pi}{2(2x+1)}\leqslant 0\right.$$

for $x \geq k$, i.e., the function $\varphi_k(x)$ is decreasing in the interval $[k, +\infty)$. Therefore, $a_{nk} \leq 0$ for $0 \leq k < n-1$.

Since the matrix $BW^{-1} = (a_{nk})$ is normal (i.e., $a_{nn} \neq 0$, $a_{nk} = 0$ for $k > n$), it is sufficient for the equivalence of this matrix to absolute convergence that [4, 5]

$$\sum_{n=0}^{\infty}|a_{nk}| = O(1),\ \varliminf_{k\to\infty}\left(|a_{kk}| - \sum_{n=k+1}^{\infty}|a_{nk}|\right) > 0.$$

We have

$$|a_{nn}| - \sum_{i=1}^{N}|a_{n+i,n}| = (2n+1)\cos n\alpha_n - |(2n+1)(\cos n\alpha_{n+1} - \cos n\alpha_n) -$$
$$- 4n\cos(n+1)\alpha_{n+1}| + \sum_{i=2}^{N}\left(\cos n\alpha_{n+i} + 2n\sin n\alpha_{n+i}\operatorname{tg}\frac{\alpha_{n+i}}{2} -\right.$$
$$\left.-\left(\cos n\alpha_{n+i-1} + 2n\sin n\alpha_{n+i-1}\operatorname{tg}\frac{\alpha_{n+i-1}}{2}\right)\right) = (2n+1)\cos n\alpha_n -$$
$$-|(2n+1)(\cos n\alpha_{n+1} - \cos n\alpha_n) - 4n\cos(n+1)\alpha_{n+1}| + \cos n\alpha_{n+N} +$$
$$+ 2n\sin n\alpha_{n+N}\operatorname{tg}\frac{\alpha_{n+N}}{2} - \left(\cos n\alpha_{n+1} + 2n\sin n\alpha_{n+1}\operatorname{tg}\frac{\alpha_{n+1}}{2}\right) \to$$
$$\to (2n+1)\cos n\alpha_n - |(2n+1)(\cos n\alpha_{n+1} - \cos n\alpha_n) - 4n\cos(n+1)\alpha_{n+1}| -$$
$$-\left(\cos n\alpha_{n+1} + 2n\sin n\alpha_n\operatorname{tg}\frac{\alpha_{n+1}}{2}\right) + 1 \quad (N \to \infty).$$

The first term on the right-hand side of the above relation converges to $\pi/2$, the second term converges to zero, and the third term converges to $\pi/2$ as $n \to \infty$. Therefore, $\varliminf_{n\to\infty}\left(|a_{nn}| - \sum_{i=n+1}^{\infty}|a_{in}|\right) = \frac{\pi}{2} - \frac{\pi}{2} + 1 = 1 > 0$. Since $a_{nn} \to \pi/2$ as $n \to \infty$, it follows from the above-obtained inequality that the following condition is fulfilled: $\sum_{n=k}^{\infty}|a_{nk}| = O(1)$. The theorem is proved.

The following theorem is proved in the same manner (on the basis of results of [2]).

THEOREM 2. The Rogosinski method is absolutely equivalent to the method of arithmetic means (the $|C_1|$ method).

Results, analogous to Theorems 1 and 2, for usual summability are contained, respectively, in [6, 7, 8].

The following lemma is proved by the method of inverse transformation in the same way as Theorem 1.

LEMMA 1. The Rogosinski – Bernstein method is absolutely equivalent to the $|C_1Z_2|$ method, where $Z_2 = (z_{nk})$ is the matrix of the Silverman – Szász method, whose elements are defined in the following manner: $z_{00} = 1$, $z_{nn} = z_{n,\,n-1} = 1/2$, $z_{nk} = 0$ $(0 \leq k < n-1,\ k > n)$.

THEOREM 3. The inclusions $|C_\alpha| \subset |B| \subset |C_\beta|$ are valid if and only if $-1 < \alpha \leq 1$, $\beta \geq 2$.

Proof. In connection with Theorem 1, it is sufficient to prove these relations for the Voron – Nörlund method defined by the sequence of the numbers $p_n = 2$ $(n > 0)$, $p_0 = 1$.