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Generalized $\varphi(Ric)$-vector fields in special pseudo-Riemannian spaces

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Abstract. The paper treats pseudo-Riemannian spaces admitting generalized $\varphi(Ric)$-vector fields. We study conditions for the existence of such vector fields in conformally flat, equidistant, reducible and Kählerian pseudo-Riemannian spaces. The obtained results can be applied for the construction of generalized $\varphi(Ric)$-vector fields distinct from $\varphi(Ric)$-vector fields. The research is carried out locally without limitations imposed on the sign of the metric tensor.

Анотація. Досліджуються псевдоріманові простори, які допускають узагальнені $\varphi(Ric)$-векторні поля. Вивчені умови існування таких векторних полів в конформно пласких, еквідістантних, звідних та келерових псевдоріманових просторах. Отримані результати можуть бути застосовані до побудови прикладів узагальнено $\varphi(Ric)$-векторних полів відмінних від $\varphi(Ric)$-векторних полів. Дослідження ведуться локально і без обмежень на знак метричного тензора.

1. Introduction

Let V_n be a pseudo-Riemannian space with a metric tensor g_{ij}. Taking into account algebraic reasoning, the papers [4, 5] introduce $\varphi(Ric)$-vector fields. Namely, those are vector fields corresponding to the equations:

$$\varphi_{i,j} = sR_{ij}, \tag{1.1}$$

where R_{ij} is the Ricci tensor, s is some constant, and comma "," designates covariant derivative by the connection V_n.

Keywords: pseudo-Riemannian spaces, $\varphi(Ric)$-vector fields, quasi-Einstein spaces
Ключові слова: псевдо-ріманові простори, $\varphi(Ric)$-векторні поля, квазі-ейнштенові простори
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Some geometric properties of these vector fields were considered by I. Hinterleitner and V. Kiosak [4, 5]. They studied conformally flat pseudo-Riemannian spaces and spaces admitting $\varphi(Ric)$ and concircular fields simultaneously.

The concept of $\varphi(Ric)$-vector fields can be extended by introducing the so-called generalized $\varphi(Ric)$-vector fields. Recall that a $\varphi(Ric)$-vector fields requires $s \equiv \text{const}$, while for generalized $\varphi(Ric)$-vector fields s is not necessarily constant but some invariant.

2. Generalized $\varphi(Ric)$-vector fields in conformally-flat spaces

For the generalized fields the integrability conditions of equations (1.1) can be written as follows:

$$\varphi_\alpha R^\alpha_{ijk} = s(R_{ij,k} - R_{ik,j}) + s_k R_{ij} - s_j R_{ik},$$

where R^h_{ijk} is the Riemann tensor chosen so that $R^h_{ijk} = g^{\alpha h} R_{\alpha ijk}$, g^{ij} are elements of inverse matrix g_{ij}, and $s_k = s_{,k}$.

Taking into account properties of the Riemann tensor R^h_{ijk}, we can write down the following expression:

$$\varphi_\alpha R^\alpha_{ijk} = s R^\alpha_{ijk,\alpha} + s_k R_{ij} - s_j R_{ik}. \tag{2.1}$$

Wrapping (2.1), we get another expression:

$$\varphi_\alpha R^\alpha_k = \tfrac{s}{2} R_{,k} + s_k R - s_\alpha R^\alpha_k,$$

where R is the scalar curvature and $R^h_i = R_{\alpha i} g^{\alpha h}$.

A *conformally flat* space is a pseudo-Riemannian space V_n satisfying the following conditions, [24]:

$$R_{hijk} = P_{hk} g_{ij} - P_{hj} g_{ik} + P_{ij} g_{hk} - P_{ik} g_{hj}, \tag{2.2}$$

$$P_{ij,k} - P_{ik,j} = 0, \tag{2.3}$$

where

$$P_{ij} = \tfrac{1}{n-2}\big(R_{ij} - \tfrac{1}{2(n-1)} R g_{ij}\big). \tag{2.4}$$

The equation (2.2) implies that

$$\varphi_\alpha R^\alpha_{ijk} = \varphi_\alpha P^\alpha_k g_{ij} - \varphi_\alpha P^\alpha_j g_{ik} + P_{ij}\varphi_k - P_{ik}\varphi_j, \tag{2.5}$$

where

$$P^h_i = g^{\alpha h} P_{i\alpha}.$$

Differentiating (2.2), wrapping it by indices h and l, and taking into account (2.3), we obtain the following:

$$R^\alpha_{ijk,\alpha} = P^\alpha_{k,\alpha} g_{ij} - P^\alpha_{j,\alpha} g_{ik},$$

which is the same as

$$R^\alpha_{ijk,\alpha} = P_{,k} g_{ij} - P_{,j} g_{ik}, \tag{2.6}$$

where
$$P_{\alpha\beta}g^{\alpha\beta} = P.$$
Substituting (2.5) and (2.6) into (2.1), we get
$$\left(\varphi_\alpha P^\alpha_k - sP_{,k}\right) g_{ij} - \left(\varphi_\alpha P^\alpha_j - sP_{,j}\right) g_{ik} + \\ + \varphi_k P_{ij} - \varphi_j P_{ik} = s_{,k}R_{ij} - s_{,j}R_{ik}.$$
Taking into account (2.4) and grouping it we obtain
$$\overset{1}{\tau}_k g_{ij} - \overset{1}{\tau}_j g_{ik} + \overset{2}{\tau}_k R_{ij} - \overset{2}{\tau}_j R_{ik} = 0, \tag{2.7}$$
where
$$\overset{1}{\tau}_i = \tfrac{1}{n-2}\varphi_\alpha R^\alpha_i - \tfrac{s}{2(n-1)} R_{,i} - \tfrac{R}{(n-1)(n-2)}\varphi_i,$$
$$\overset{2}{\tau}_i = \tfrac{1}{n-2}\varphi_i - s_{,i}.$$
Finally, wrapping (2.7) we get
$$(n-1)\,\overset{1}{\tau}_k + R\,\overset{2}{\tau}_k - \overset{2}{\tau}_\alpha R^\alpha_k = 0.$$

Let us multiply (2.7) by a vector $\overset{2}{\tau}{}^i = g^{\alpha i}\,\overset{2}{\tau}_\alpha$ and wrap it by index i:
$$(n-2)\left(\overset{1}{\tau}_k\,\overset{2}{\tau}_j - \overset{1}{\tau}_j\,\overset{2}{\tau}_k\right) = 0.$$
If $\overset{2}{\tau}_i \neq 0$, then we can choose a vector ξ^i so that $\xi^\alpha\,\overset{2}{\tau}_\alpha = 1$. Then
$$\overset{1}{\tau}_i = \rho\,\overset{2}{\tau}_i, \tag{2.8}$$
where ρ is some invariant.

After substitution of (2.8) the equation (2.7) reduces to the following one:
$$\overset{2}{\tau}_k(R_{ij} + \rho g_{ij}) - \overset{2}{\tau}_j(R_{ik} + \rho g_{ik}) = 0. \tag{2.9}$$
Multiplying (2.9) by ξ^k we obtain
$$R_{ij} + \rho g_{ij} - \overset{2}{\tau}_j(R_{\alpha i} + \rho g_{\alpha i})\xi^\alpha = 0. \tag{2.10}$$
Due to the symmetry of tensors (2.10) can be re-written as follows:
$$R_{ij} + \rho g_{ij} = u_i u_j, \tag{2.11}$$
which implies
$$\rho = \tfrac{1}{n}u_\alpha u^\alpha - \tfrac{R}{n}.$$
Hence we get from (2.11) that
$$R_{ij} - \tfrac{R}{n}g_{ij} = u_i u_j - \tfrac{1}{n}u_\alpha u^\alpha g_{ij},$$

which is equivalent to the identity

$$E_{ij} = u_i u_j - \tfrac{1}{n} u_\alpha u^\alpha g_{ij}, \tag{2.12}$$

where $E_{ij} = R_{ij} - \frac{R}{n} g_{ij}$ is the Einstein tensor.

Equations (2.12) are characteristic for quasi-Einstein spaces, [13, 14]. Also the equations (2.11) under conditions (2.2) led us to the spaces of quasi-constant curvature, [10].

Consider the case when $\overset{2}{\tau}_i = 0$, that is

$$s_{,i} = \tfrac{1}{n-2} \varphi_i.$$

Then the equation (2.8) implies that $\overset{1}{\tau}_i = 0$, which means that

$$\tfrac{1}{n-2} \varphi_\alpha R_i^\alpha - \tfrac{s}{2(n-1)} R_{,i} - \tfrac{R}{(n-1)(n-2)} \varphi_i = 0.$$

Thus, we get the following theorem:

Theorem 2.1. *If a conformally flat pseudo-Riemannian space V_n admits a generalized $\varphi(Ric)$-vector field such that $\varphi_i \neq (n-2)s_{,i}$, then this space is a quasi-Einstein space of quasi-constant curvature.*

Note, that if in the equation (2.12) a vector u_i is gradient, namely

$$u_i = u_{,i} = \partial_i u,$$

then this pseudo-Riemannian space is a subprojective Kagan space, [7]. It is well-known that subprojective Kagan spaces are equidistant. Therefore in the subsequent sections we will consider equidistant spaces admitting generalized $\varphi(Ric)$-vector fields.

3. Generalized $\varphi(Ric)$-vector spaces in equidistant spaces

A pseudo-Riemannian space V_n with a metric tensor g_{ij} is called *equidistant* whenever it admits a vector field $\psi_i \neq 0$ corresponding to equations

$$\psi_{i,j} = \tau g_{ij}, \tag{3.1}$$

where τ is some invariant. For $\tau \neq 0$, the space V_n is called an equidistant space of *main* type, while for $\tau = 0$ it is a space of *special* type, [24, 26].

Following by K. Yano we will say that a vector field complying with the conditions (3.1) is *concircular*.

Integrability conditions for the main equations (3.1) can be written as follows:

$$\psi_\alpha R^\alpha_{ijk} = g_{ij} \tau_{,k} - g_{ik} \tau_{,j}. \tag{3.2}$$

They imply that

$$\tau_{,i} = \frac{1}{n-1} \psi_\alpha R_i^\alpha. \tag{3.3}$$

Notice that the set of equations (3.1) and (3.3) is closed. It is a system of linear differential equations in covariant derivatives of the first order of Cauchy type with coefficients unequivocally defined by the space V_n, with respect to an unknown vector ψ_i and an invariant τ.

Equidistant spaces constitute an important part of theory of geodesic mappings. They are crucial in the theory of modelling with preservation of geodesic lines, [11].

The integrability conditions (3.1) imply that

$$\tau_{,k} = B\psi_k, \tag{3.4}$$

where B is some invariant. Taking into account (3.2) and (3.3) we then get

$$\begin{aligned} \psi_\alpha R^\alpha_{ijk} &= B(\psi_k g_{ij} - \psi_j g_{ik}), \\ \psi_\alpha R^\alpha_i &= (n-1)B\psi_i. \end{aligned} \tag{3.5}$$

Differentiating (3.4) and taking into account (3.1) we obtain that

$$\tau R_{hijk} + \psi_\alpha R^\alpha_{ijk,h} = B_{,h}(\psi_k g_{ij} - \psi_j g_{ik}) + B\tau(g_{hk}g_{ij} - g_{jh}g_{ik}). \tag{3.6}$$

Cycle (3.6) by indices h, j, k:

$$B_{,h}(\psi_k g_{ij} - \psi_j g_{ik}) + B_{,j}(\psi_h g_{ik} - \psi_k g_{ih}) + B_{,k}(\psi_j g_{ih} - \psi_h g_{ij}) = 0,$$

and wrap by indices i, j:

$$B_{,h}\psi_k - B_{,k}\psi_h = 0.$$

Multiplying the latter identity by the vector ξ^k chosen so that $\psi_\alpha \xi^\alpha = 1$ and $B_{,\alpha}\xi^\alpha = A$, we obtain that

$$B_{,h} = A\psi_h.$$

Then equation (3.6) transforms into the following one:

$$\tau R_{hijk} + \psi_\alpha R^\alpha_{ijk,h} = A(\psi_h\psi_k g_{ij} - \psi_h\psi_j g_{ik}) + B\tau(g_{hk}g_{ij} - g_{jh}g_{ik}).$$

Wrapping it by indices h, k we get that

$$\tau R_{ij} + \psi^\alpha R^\beta_{ji\alpha,\beta} = (A\psi_\alpha\psi^\alpha + 3B\tau)g_{ij} - A\psi_i\psi_j. \tag{3.7}$$

Multiplying further (3.7) by an invariant s and taking into account (2.1) we obtain:

$$\begin{aligned} &\psi^\alpha \varphi_\beta R^\beta_{ji\alpha} - \psi^\alpha s_\alpha R^\beta_{ji} + \psi^\alpha R_{\alpha j} s_i = \\ &\qquad = s(A\psi_\alpha\psi^\alpha + 3B\tau)g_{ij} - As\psi_i\psi_j - \tau s R_{ij}. \end{aligned} \tag{3.8}$$

Let us rewrite (3.5) as follows:

$$\psi^\alpha R_{kji\alpha} = B(\psi_k g_{ij} - \psi_j g_{ik}).$$

Multiplying the latter identity by φ^k, then wrapping by index k, and further substituting the result into (3.8) we will get:

$$B\psi_\alpha\varphi^\alpha g_{ij} - \psi_j\psi_i - \psi^\alpha s_\alpha R_{ji} + \psi^\alpha R_{\alpha j}s_i = \\ = s(A\psi_\alpha\psi^\alpha + 3B\tau)g_{ij} - As\psi_i\psi_j - \tau s R_{ij}. \tag{3.9}$$

Alternating (3.9) by indices i and j we obtain

$$\psi_j\left((n-1)Bs_i - \varphi_i\right) - \psi_i\left((n-1)Bs_j - \varphi_j\right) = 0,$$

which implies

$$(n-1)Bs_i = \varphi_i + \nu\psi_i, \tag{3.10}$$

where ν is some invariant.

Then (3.9) reduces to the following identity:

$$\left(B\psi_\alpha\varphi^\alpha - sA\psi_\alpha\psi^\alpha - 3sB\tau\right)g_{ij} = \left(\psi^\alpha s_\alpha - \tau s\right)R_{ij} - (As-\nu)\psi_i\psi_j. \tag{3.11}$$

Wrapping (3.11), we get that

$$B\psi_\alpha\varphi^\alpha - sA\psi_\alpha\psi^\alpha - 3sB\tau = \tfrac{R}{n}\left(\psi^\alpha s_\alpha - \tau s\right) - \tfrac{\psi_\alpha\psi^\alpha}{n}(As-\nu).$$

Then (3.11) can be rewritten as follows:

$$\left(\psi^\alpha s_\alpha - \tau s\right)\left(R_{ij} - \tfrac{R}{n}g_{ij}\right) - (As-\nu)\left(\tfrac{\psi_\alpha\psi^\alpha}{n}g_{ij} - \psi_i\psi_j\right) = 0,$$

which is the same as

$$\overset{3}{\tau}E_{ij} + \overset{4}{\tau}\left(\psi_i\psi_j - \tfrac{\psi_\alpha\psi^\alpha}{n}g_{ij}\right) = 0, \tag{3.12}$$

where $\overset{3}{\tau} = \psi^\alpha s_\alpha - \tau s$, $\overset{4}{\tau} = As - \nu$.

Thus, we get the following result:

Theorem 3.1. *If a equidistant pseudo-Riemannian space V_n admits generalized $\varphi(Ric)$-vector fields, then in this space the conditions* (3.10) *hold for a vector φ_i and conditions* (3.12) *for the Einstein tensor.*

Theorem 3.1 agrees well with the results of [12, 17, 18], when the latter are widened by application of the concept of the generalized $\varphi(Ric)$-vector fields.

4. Generalized $\varphi(Ric)$-vector fields in reducible fields

A pseudo-Riemannian space V_n with a metric tensor g_{ij} is called *locally reducible*, whenever at each point M of V_n there are local coordinates $y^1, y^2, \dots, y^n$ in which the main matrix is of the following form:

$$I = g_{pq}(y^r)dy^p dy^q + g_{\sigma\mu}(y^\nu)dy^\sigma dy^\mu, \tag{4.1}$$

$$(p, q, r = 1, 2, \dots, m,\ \sigma, \mu, \nu = m+1, m+2, \dots, n),$$

where g_{pq} depend only on the variables $y^1, y^2, \ldots, y^m$, and $g_{\sigma\mu}$ depends only on $y^{m+1}, y^{m+2}, \ldots, y^n$.

In what follows, a locally reducible space will be called reducible. Thus, a reducible pseudo-Riemannian space $V_n(g_{ij})$, by definition, is a product of two pseudo-Riemannian spaces $\overset{1}{V}_m(g_{pq})$ and $\overset{2}{V}_{n-m}(g_{\sigma\mu})$:

$$g_{ij} = \begin{pmatrix} g_{pq} & | & 0 \\ - & & - \\ 0 & | & g_{\sigma\mu} \end{pmatrix}$$

Each of $\overset{1}{V}_m$ and $\overset{2}{V}_{n-m}$ can be either reducible or not whence (4.1) can be re-written as follows:

$$ds^2 = \sum_{k=1}^{r} ds_k^2 \quad (r > 1),$$

where ds_k^2 is the quadratic form of V_{m_k}, $(m_1 + m_2 + \ldots + m_n = n)$.

For a given pseudo-Riemannian space V_n a number r can take different values. The maximal value of r is called the *mobility* of a pseudo-Riemannian space in respect to its reduction.

A pseudo-Riemannian space V_n is reducible if and only if it contains a symmetric tensor $a_{ij} = cg_{ij}$ (in case of some constant c), which complies to the following conditions:

$$a_{i\alpha} a_j^\alpha = a_{ij}, \tag{4.2}$$

$$a_{ij,k} = 0, \tag{4.3}$$

where $a_j^i = a_{\alpha j} g^{\alpha i}$.

Equations (4.2) and (4.3) is an invariant (with respect to the given local coordinates) condition, being necessary and sufficient for a pseudo-Riemannian space V_n to be reducible. In the above-mentioned form it was formulated by P. A. Shirokov [23].

A tensor a_{ij}, that complies to condition (4.2), is called *idempotent*, while a tensor a_{ij} that complies to (4.3) is a covariant constant tensor.

The requirement of idempotency can be replaced by the requirement that the matrix of a tensor a_{ij} should have simple elementary divisors and real roots (as was proved by G. Kruchkovich [20]). A similar formulation of this characteristic can be found as an exercise in the book of L. P. Eisenhart “Riemannian geometry”, [2], however without an addition on the real roots. Without that requirement, it is easy to see that the characteristic is wrong.

Taking into account the Ricci identity, integrability conditions for the equation (4.3) can be formulated as follows

$$a_{\alpha i} R_{jkl}^\alpha + a_{\alpha j} R_{ikl}^\alpha = 0. \tag{4.4}$$

Cycling the latter by $(i,\ k,\ l)$, we obtain

$$a_{\alpha i}R^{\alpha}_{jkl} + a_{\alpha k}R^{\alpha}_{jli} + a_{\alpha l}R^{\alpha}_{jik} = 0.$$

Wrapping by indices $(j,\ k)$, we can write down the following expression

$$a_{\alpha i}R^{\alpha}_{l} - a_{\alpha l}R^{\alpha}_{i} = 0,$$

where

$$R^{i}_{j} = R_{\alpha j}g^{\alpha i}.$$

Tensors a_{ij} and b_{ij} satisfying the following identity

$$a^{\alpha}_{i}b_{\alpha j} = a^{\alpha}_{j}b_{\alpha i},$$

are said to commute, [1].

Theorem 4.1. *Each reducible pseudo-Riemannian space V_n contains an idempotent tensor which commutes with the Ricci tensor of V_n.*

Differential extensions of (4.4) can be written as follows:

$$a_{\alpha i}R^{\alpha}_{jkl,m} + a_{\alpha j}R^{\alpha}_{ikl,m} = 0.$$

Wrapping the latter by indices l and m and taking into account (2.1) and (4.4), we get

$$a^{\alpha}_{i}s_{\alpha}R_{jk} - s_{j}a^{\alpha}_{i}R_{\alpha k} + a^{\alpha}_{j}s_{\alpha}R_{ik} - s_{i}a^{\alpha}_{j}R_{\alpha k} = 0. \tag{4.5}$$

Alternating by indices j and k we obtain

$$a^{\alpha}_{i}s_{\alpha}R_{jk} - a^{\alpha}_{k}s_{\alpha}R_{ji} - s_{i}a^{\alpha}_{j}R_{\alpha k} + s_{k}a^{\alpha}_{j}R_{\alpha i} = 0.$$

Let us re-assign indices i and k:

$$a^{\alpha}_{i}s_{\alpha}R_{kj} - a^{\alpha}_{j}s_{\alpha}R_{ki} - s_{i}a^{\alpha}_{k}R_{\alpha j} + s_{j}a^{\alpha}_{k}R_{\alpha i} = 0. \tag{4.6}$$

Adding up the equations (4.6) and (4.5), we arrive at

$$a^{\alpha}_{i}s_{\alpha}R_{jk} - s_{i}a^{\alpha}_{j}R_{\alpha k} = 0.$$

If $s_i \not\equiv 0$, then

$$a^{\alpha}_{j}R_{\alpha k} = uR_{jk}, \tag{4.7}$$

where u is some invariant.

Multiplying (4.7) by $a^{\alpha}_{i} = a_{\alpha i}g^{\alpha j}$ and taking into account (4.2) and (4.7) we get

$$a^{\alpha}_{j}R_{\alpha k} = u^{2}R_{jk}. \tag{4.8}$$

Substracting (4.8) out of (4.7) we can see that $u = 0$ or $u = 1$ and

$$a^{\alpha}_{i}s_{\alpha} = us_{i}. \tag{4.9}$$

Thus, we have proved the following theorem:

Theorem 4.2. *In a reducible pseudo-Riemannian space admitting a generalized $\varphi(Ric)$-vector field conditions* (4.7) *and* (4.9) *hold true with $u = 0$ or $u = 1$.*

5. Generalized $\varphi(Ric)$-vector fields in Kählerian spaces

A Kähler space K_n, $(n = 2N)$, is a pseudo-Riemannian space with a metric tensor $g_{ij}(x)$ containing structure $F_i^h(x)$, which satisfies the following condition, [3, 6, 19]:

$$F_\alpha^h F_i^\alpha = -\delta_i^h; \qquad F_{(ij)} = 0; \qquad F_{i,j}^h = 0, \tag{5.1}$$

where $F_{ij} \equiv g_{i\alpha} F_j^\alpha$, comma denotes a covariant derivative by the connection of K_n, and brackets (ij) mean symmetrizing by indices i, j.

Notice that Kähler spaces were introduced by P. A. Shirokov under the name of A-spaces. Later these spaces were studied by E. Kähler and in literature they are basically known as Kähler spaces, [6, 8].

For convenience define an operation of conjugation in K_n as follows:

$$A_{\bar{i}\dots}^{\;\dots} \equiv A_{\alpha\dots}^{\;\dots} F_i^\alpha, \qquad\qquad B_{\;\dots}^{\bar{i}\dots} \equiv B_{\;\dots}^{\alpha\dots} F_\alpha^i, \tag{5.2}$$

where A and B are some tensors of any valence. The equations (5.1) and (5.2) imply the following properties:

$$\begin{aligned} A_{\bar{\bar{i}}} &= -A_i, & B^{\bar{\bar{i}}} &= -B^i, \\ A_{\bar\alpha} B^\alpha &= A_\alpha B^{\bar\alpha}, & A_{\bar\alpha} B^{\bar\alpha} &= -A_\alpha B^\alpha, \\ (A_{\bar i})_{,\,j} &= A_{\bar i\,,\,j}, & (B^{\bar i})_{,\,j} &= B^{\bar i}{}_{,\,j}. \end{aligned}$$

A metric tensor and Kronecker symbols satisfy the conditions:

$$g_{\bar i\,\bar j} = g_{ij}, \qquad g_{\bar i j} = -g_{i\bar j}, \qquad \delta_{\bar i}^h = \delta_i^{\bar h} = F_i^h, \qquad \delta_{\bar i}^{\bar h} = -\delta_i^h.$$

But the well-known identities, Ricci tensor and Riemannian tensor have the following additional properties:

$$R_{\bar h \bar i jk} = R_{hijk}; \qquad R_{\bar\alpha jk}^\alpha = 2R_{j\bar k}, \qquad R_{\bar i\,\bar j} = R_{ij}.$$

The inner objects of K_n are objects defined by the metric tensor g_{ij} and the structure F_i^h.

Consider Kähler spaces admitting generalized $\varphi(Ric)$-vector fields. Applying the operation of conjugation by indices j and k to (2.1) we get

$$\varphi_\alpha R_{ijk}^\alpha = s R_{ijk,\alpha}^\alpha + s_{\bar k} R_{i\bar j} - s_{\bar j} R_{i\bar k}. \tag{5.3}$$

Subtracting (5.3) out of (2.1) gives:

$$s_k R_{ij} - s_j R_{ik} - s_{\bar k} R_{i\bar j} + s_{\bar j} R_{i\bar k} = 0. \tag{5.4}$$

Wrapping further by indices i and j, we obtain

$$Rs_k = 2s_\alpha R^\alpha_k.$$

Finally, mltiplying (5.4) by a vector s^k and wrapping by an index k we get the following identity:

$$s^\alpha s_\alpha R_{ij} = \tfrac{R}{2}\left(s_i s_j + s_{\bar{j}} s_{\bar{i}}\right). \tag{5.5}$$

Thus, we proved the following theorem:

Theorem 5.1. *If a Kähler space admits generalized $\varphi(Ric)$-vector fields, then the conditions* (5.5) *hold true.*

Note that in the case defined by (5.5) the important role is played by a vector s_i, and, this is the reason why they do carry information on generalized $\varphi(Ric)$-vector fields for the case when s_i is not constant.

6. Conclusions

The notion of generalized $\varphi(Ric)$-vector field extends the concept of the class of $\varphi(Ric)$-vector fields in pseudo-Riemannian spaces. This generalization is achieved by omitting the requirement imposed on the coefficient of proportionality which now is not required to be constant. The paper [25] studies non-trivial geodesic mappings of pseudo-Riemannian spaces admitting $\varphi(Ric)$-vector fields.

Special pseudo-Riemannian spaces admitting generalized $\varphi(Ric)$-vector fields require future study. In particular it is necessary to understand which properties of such spaces are shared with spaces admitting $\varphi(Ric)$-vector fields and which properties are new. Those spaces can find applications in other fields of mathematics, [9, 21, 22, 27].

In the present paper we found necessary conditions on Ricci tensor for conformally flat, equidistant, reducible and Kähler pseudo-Riemannian spaces to admit generalized $\varphi(Ric)$-vector fields. It turns out that in all the cases those pseudo-Riemannian spaces are quasi-Einstein pseudo-Riemannian spaces. Geodesic and conformal mappings of these spaces are studied in [12, 15, 16]. The obtained results can be applied for the construction and further research on geometric properties of these spaces.

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