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Holomorphically Projective Mappings of Special Kähler Manifolds

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Abstract. The paper treats diffeomorphisms of special Kählerian manifolds, which preserve analytical planar curves. The research is conducted locally, in tensor shape, without limitations on the sign of metric. The problem is proved to be equivalent to solving a system of differential equations in covariant derivatives.

SPECIAL QUASI-EINSTEIN SPACES

Pseudo-Riemannian space V_n with a metric tensor g_{ij} is called an Einstein space, if the following conditions are true for the former:

$$E_{ij} = 0 \tag{1}$$

where E_{ij} – Einstein tensor,

$$E_{ij} = R_{ij} - \frac{R}{n} g_{ij},$$

R_{ij} – Ricci tensor, R – scalar curvature of spaces V_n.

The Einstein spaces can be generalized by several ways. In this context the spaces with the Einstein tensor that differs from zero in a certain way are of particular interest.

$$E_{ij} - D_{ij} = 0 \tag{2}$$

here D_{ij} is a tensor, that will be called a defect of Einstein tensor.

The defect of Einstein tensor bears certain algebraic and differential limitations which arise from needs of physics.

The notion of quasi-Einstein space seems to be applied for the first time denoting the spaces where:

$$D_{ij} = R_{\alpha i} R^{\alpha}_{j} - R_{\alpha\beta} R^{\alpha\beta}_{.ij.} \tag{3}$$

here $R^i_j = g^{\alpha i} R_{\alpha j}$, $R^{hk}_{ij} = R_{\alpha ij\beta} g^{\beta k}$, R_{ijkl} – Riemannian tensor V_n, g^{ij} – elements of reverse matrix for metric tensor g_{ij} [1].

These spaces will be named quasi-Einstein spaces of the first type here.

Let us treat pseudo-Riemannian spaces, where the following conditions are true

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$$R_{\alpha i}R^{\alpha}_{jkl} + R_{\alpha j}R^{\alpha}_{ikl} = A_{li}R_{jk} + A_{lj}R_{ik} - A_{ki}R_{jl} - A_{kj}R_{il} \quad (4)$$

where A_{ij} - some symmetrical tensor.

The following lemma is true.

Lemma 1. *If pseudo-Riemannian space V_n with scalar curvature that differs from zero is a quasi-Einstein space of the first type and the following conditions (4) are true for the former then the equations (4) can be written as*

$$R_{\alpha i}R^{\alpha}_{jkl} + R_{\alpha j}R^{\alpha}_{ikl} = \frac{1}{n}(g_{li}R_{jk} + g_{lj}R_{ik} - g_{ki}R_{jl} - g_{kj}R_{il}). \quad (5)$$

We will designate these spaces as special quasi-Einstein spaces of the first type and denote them by $KPE(I)$.

KÄHLERIAN SPACES

In order to study the physical reality, modeling with the help of pseudo-Riemannian spaces is a useful and powerful tool. In the course of such a research there is always a need to estimate which properties of proto-types are preserved in a model. The answer to this question lays in the array of mappings investigations. While referring to complex physical fields, researchers are forced to pay attention to particular pseudo-Riemannian spaces and among the latter – Kählerian spaces.

Here we treat the objects commonly known as Kählerian spaces (1933) [2], however they were introduced as early as 1925 by P.A. Shirokov. A.P. Shirokov proved the equivalence of A-spaces of P.A. Shirokov and Kählerian spaces [3].

Kählerian space is an even-dimensional pseudo-Riemannian space K_n $(n = 2m > 2)$ with metric tensor a_{ij}, where affinor F_i^h exists and satisfies the conditions:

$$F_{\alpha}^{h}F_{i}^{\alpha} = -\delta_i^h, \qquad F_{hi,i} = 0$$
$$F_{hi} + F_{ih} = 0 \quad (6)$$

Here $F_{ij} \overset{d}{=} g_{\alpha i}F_j^{\alpha}$, “coma” is a sign of covariant derivative, and δ_i^h – Kronecker's symbols.

Let us treat the special quasi-Einstein spaces of the first type. We will multiply (5) by $F_{k`}^{k} \cdot F_{l`}^{l}$, wrapping by indexes k and l, and taking into account the properties of Kählerian spaces, then we will subtract the result from (5). It will permit us to proof the following theorem.

Theorem 1. *There is no special Kählerian quasi-Einstein spaces of the first type that are not Einstein spaces.*

Thus, taking into account the properties of Kählerian spaces, let us introduce spaces where the following conditions are true

$$R_{\alpha i}R^{\alpha}_{jkl} + R_{\alpha j}R^{\alpha}_{ikl} = \frac{1}{n}(R_{li}g_{jk} + R_{lj}g_{ik} - R_{ki}g_{jl} - R_{kj}g_{il}) +$$
$$+\frac{1}{n}(R_{\alpha i}g_{j\beta} + R_{\alpha j}g_{i\beta} - R_{\beta i}g_{j\alpha} - R_{\beta j}g_{i\alpha})F_l^{\alpha}F_k^{\beta} \quad (7)$$

These spaces will be called widen special quasi-Einstein spaces of the first type and will be denoted by $KPE^*(I)$

HOLOMORPHICALLY PROJECTIVE MAPPINGS $KPE^*(I)$

Analytical planar curve L for a Kählerian space is a curve described by an equation $x^h = x^h(t)$, such that the following conditions are true:

$$\frac{d\xi^h}{dt} + \Gamma^h_{\alpha\beta}\xi^\alpha\xi^\beta = \rho_1(t)\xi^h + \rho_2(t)F^h_\alpha\xi^\alpha, \tag{8}$$

where $\xi^h \equiv \frac{dx^h}{dt}$, ρ_1, ρ_2 - are functions of the argument t.

Diffeomorphism γ between points of Kählerian spaces K_n and $\overline{K}_n$ is called holomorphically projective mapping, if every analytically planar curve K_n is mapped into analytically planar curve $\overline{K}_n$ [4,5,6].

A necessary and sufficient condition for the Kahlerian space to permit non-trivially holomorphical mappings of K_n onto $\overline{K}_n$ in their general system of coordinates is the fulfillment of the relations:

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i\psi_j + \delta^h_j\psi_i + \delta^h_\alpha\psi_\beta F^\alpha_i F^\beta_j + \delta^h_\alpha\psi_\beta F^\alpha_j F^\beta_i,$$
$$\overline{F}^h_i = F^h_i, \tag{9}$$

where $\psi_i \equiv \psi_{,i}$.

It is necessary and sufficient for the Kählerian space K_n in order to permit non-trivial holomorphically projective mappings that the system of differential equations

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} + \lambda_\alpha g_{\beta k}F^\alpha_i F^\beta_j + \lambda_\alpha g_{\beta k}F^\alpha_j F^\beta_i; \tag{10}$$

$$n\lambda_{i,j} = \mu g_{ij} - a_{\alpha\beta}R^{\alpha\beta}_{.ij.} + a_{\alpha i}R^\alpha_j; \tag{11}$$

$$\mu_i = 2\lambda_\alpha R^\alpha_i \tag{12}$$

had non-trivial solution according to tensor $a_{ij} = a_{ji} = a_{\alpha\beta}F^\alpha_j F^\beta_i$, vector $\lambda_i \neq 0$ and invariant μ [3].

Let us treat holomorphically projective mappings of Kählerian spaces $KPE^*(I)$.

The theorem can be proved.

Theorem 2. *If Kählerian spaces $KPE^*(I)$ permit non-trivial holomorphically projective mappings, then equations (11), (12) for them assume the form of*

$$\lambda_{i,j} = \mu g_{ij} + Ba_{ij} \tag{13}$$

$$\mu_i = 2B\lambda_i \tag{14}$$

where B – a constant.

Proof.

Conditions of integrability (taking into account Ricci identities) are as follows for the equations (10)

$$a_{\alpha i}R^\alpha_{jkl} + a_{\alpha j}R^\alpha_{ikl} = \lambda_{l,i}g_{jk} + \lambda_{l,j}g_{ik} - \lambda_{k,i}g_{jl} - \lambda_{k,j}g_{il} +$$
$$+(\lambda_{l,\alpha}g_{\beta k} + \lambda_{l,\beta}g_{\alpha k} - \lambda_{k,\alpha}g_{\beta l} - \lambda_{k,\beta}g_{\alpha l})F^\alpha_i F^\beta_j. \tag{15}$$

Multiplying (7) by R^l_m, summing up by indexes l, let us sum up by indexes k and m. Taking into account the latter, we will obtain

$$R^\alpha_m\lambda_{\alpha,i}g_{jk} + R^\alpha_m\lambda_{\alpha,j}g_{ik} - \lambda_{k,i}R_{jm} - \lambda_{k,j}R_{im} + R^\alpha_k\lambda_{\alpha,i}g_{jm} + R^\alpha_k\lambda_{\alpha,j}g_{im} -$$

$$-\lambda_{m,i}R_{jk}-\lambda_{m,j}R_{ik}+(R_m^\alpha\lambda_{\alpha,\beta}g_{\gamma k}+R_m^\alpha\lambda_{\alpha,\gamma}g_{\beta k}-\lambda_{k,\beta}R_{\gamma m}-\lambda_{k,\gamma}R_{\beta m}+$$
$$+R_k^\alpha\lambda_{\alpha,\beta}g_{\gamma m}+R_k^\alpha\lambda_{\alpha,\gamma}g_{\beta m}-\lambda_{m,\beta}R_{\gamma k}-\lambda_{m,\gamma}R_{\beta k})F_i^\beta F_j^\gamma=$$
$$(16)$$
$$=\frac{1}{n}(R_m^\alpha a_{\alpha i}g_{jk}+R_m^\alpha a_{\alpha j}g_{ik}-a_{ki}R_{jm}-a_{kj}R_{im}+R_k^\alpha a_{\alpha i}g_{jm}+R_k^\alpha a_{\alpha j}g_{im}-$$
$$-a_{mi}R_{jk}-a_{mj}R_{ik})+\frac{1}{n}(R_\beta^\alpha a_{\alpha i}g_{j\gamma}+R_\beta^\alpha a_{\alpha j}g_{i\gamma}-a_{\gamma i}R_{jm}-a_{\gamma j}R_{i\beta}+$$
$$+R_\gamma^\alpha a_{\alpha i}g_{j\beta}+R_\gamma^\alpha a_{\alpha j}g_{i\beta}-a_{\beta i}R_{j\gamma}-a_{\beta j}R_{i\gamma})F_m^\beta F_k^\gamma.$$

Wrapping up (16), in order to obtain

$$\lambda_{\alpha,i}R_j^\alpha=\lambda_{\alpha,j}R_i^\alpha \tag{17}$$

Multiplying (16) by $F_l^i F_q^m$, after transformations we will see that the theorem 2 is true [7,8,9].

Let us define the constant B for quasi-Einstein spaces satisfying the equations (3).

Conditions of integrability for equations (13) are as follows:

$$\lambda_\alpha R_{ijk}^\alpha=B(\lambda_k g_{ji}-\lambda_j g_{ki}+\lambda_\alpha g_{\beta i}-\lambda_\beta g_{\alpha i})F_k^\alpha F_j^\beta; \tag{18}$$

$$\lambda_\alpha R_k^\alpha=Bn\lambda_k. \tag{19}$$

Multiplying (3) by vector λ^i and wrapping, we will obtain

$$(Bn-\frac{R}{n})(Bn-1)=0 \tag{20}$$

Thus, the theorem is true.

Theorem 3. *If the conditions (3) and (13) are true, then constant B equals to:*

$$B=\frac{1}{n} \tag{21}$$

or

$$B=\frac{R}{n^2}. \tag{22}$$

Let us perform the following substitutions into eq. (13):

$$a_{ij}=e^{2\varphi}\overline{g}^{\alpha\beta}g_{\alpha i}g_{\beta j}; \tag{23}$$

$$\lambda_i=-e^{2\varphi}\overline{g}^{\alpha\beta}g_{\alpha i}g_{\beta,} \tag{24}$$

Then it is evident that:

$$\overline{B}\overline{g}_{ij}=Bg_{ij}+\varphi_{ij}, \tag{25}$$

where

$$\varphi_{ij}=\varphi_{i,j}-\varphi_i\varphi_j+\varphi_\alpha\varphi_\beta F_i^\alpha F_j^\beta.$$

Taking into account that in course of holomorphically projective mappings the respective objects are connected by relationships:

$$\overline{R}_{ijk}^h=R_{ijk}^h+\delta_k^h\psi_{ji}-\delta_k^h\psi_{ji}+$$
$$(26)$$
$$+(\delta_\alpha^h\psi_{\beta i}-\delta_\beta^h\psi_{\alpha i})F_k^\alpha F_j^\beta+2\delta_\alpha^h\psi_{\beta k}F_i^\alpha F_j^\beta;$$

$$\overline{R}_{ij}=R_{ij}+(n+2)\psi_{ij} \tag{27}$$

and substituting with (25), we will obtain:

$$\overline{R}_{ijk}^h=R_{ijk}^h+$$

$$+\overline{B}(\delta_k^h \overline{g}_{ij} - \delta_j^h \overline{g}_{ik} + (\delta_\alpha^h \overline{g}_{\beta i} - \delta_\beta^h \overline{g}_{\alpha i}) F_k^\alpha F_j^\beta + 2\delta_\alpha^h \overline{g}_{\beta k} F_i^\alpha F_j^\beta) - \tag{28}$$

$$- B(\delta_k^h g_{ij} - \delta_j^h g_{ik} + (\delta_\alpha^h g_{\beta i} - \delta_\beta^h g_{\alpha i}) F_k^\alpha F_j^\beta + 2\delta_\alpha^h g_{\beta k} F_i^\alpha F_j^\beta)$$

$$\overline{R}_{ij} = R_{ij} + \overline{B}(n+2)\overline{g}_{ij} - B(n+2) g_{ij}. \tag{29}$$

Then, turning our attention to (28), (29) and (7), we will see that the theorem is proved.

If a special space permits a mapping on spaces of the same type then this type of spaces is called *closed in relation to* the above mentioned mapping.

Theorem 4. *Kählerian spaces* $KPE^*(I)$ *are closed in relation to holomorphically projective mappings.*

Therefore, when modeling Kählerian spaces with preservation of analytical planar curves, if prototype is $KPE^*(I)$ then the model is also a $KPE^*(I)$.

Let us find covariant derivative for (25), taking into account the fact that $\overline{g}_{ij}$ is a metric tensor of $\overline{K}_n$. Then, we will obtain:

$$\varphi_{i,jk} = B(2\varphi_k g_{ij} + \varphi_i g_{jk} + \varphi_j g_{ik} + \varphi_\alpha g_{\beta k} + \\ + \varphi_\beta g_{\alpha k} + 4\varphi_k \varphi_i \varphi_j + 4\varphi_k \varphi_\alpha \varphi_\beta) F_i^\alpha F_j^\beta. \tag{30}$$

Applying to the latter methods developed in [7,10] we can convincingly treat the properties of $KPE^*(I)$ spaces "in general." In particular, taking into account the expression (30) we can conclude that every closed $KPE^*(I)$ space is a space with constant holomorphically projective curvature.

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