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Probability measure monad on the category of fuzzy ultrametric spaces

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Abstract. It is proved that the probability measure functor comprises a monad on the category of fuzzy ultrametric spaces and nonexpanding maps. It is also proved that the G-symmetric power functor admits an extension on the Kleisli category of this monad (i.e. the category of fuzzy ultrametric spaces and nonexpanding measure-valued maps).

Key Words and Phrases: Fuzzy ultrametric, Probability measure, Monad, Extension of functors

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1. Introduction

The natural generalization of metric spaces can be obtained, if the metric takes its values not in the set of real numbers, but in some other set. If this other set is that of probability distributions, we come to the notion of probabilistic metric space [9]. Close to this concept is that of Menger probabilistic metric spaces. As in probabilistic metric spaces, the distance between points x, y in the probabilistic metric Menger spaces is a distribution function F_{xy}. By axiomatizing properties of the mapping $(x, y, t) \mapsto F_{xy}(t)$, different authors came to the concept of fuzzy metric space. In this article we use the term fuzzy metric space in the sense of paper [2]. One of the motivations is the fact is that such fuzzy metric spaces generate the natural topology, which is metrizable. The theory of fuzzy metric spaces looks fundamentally richer than the theory of metric spaces - and this is natural, since any fuzzy metric is a function of three variables. Some phenomena occurring in the theory of fuzzy metric spaces do not have metric counterparts. As an example, mention the existence of noncompletable fuzzy metric spaces (see [3]). Note also, that even for finite sets X the space of fuzzy metrics on the space X is infinite-dimensional, unlike the space of metrics - the latter is a finite cone.

In [8] the authors considered the fuzzy ultrametrization of the set of probability measures on fuzzy ultrametric spaces. The aim of this paper is to show that the obtained probability measure functor in the category of fuzzy ultrametric spaces and nonexpanding maps determines a monad on this category. The Kleisli category of this monad is that

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of fuzzy ultrametric spaces and measure-valued maps. It is proved that the G-symmetric power functor SP^n_G admits an extension onto the latter category. The natural transformations of the symmetric power functors in the category of fuzzy ultrametric spaces and nonexpanding maps are also the natural transformations of the symmetric power functors in the Kleisli category.

2. Preliminaries

Recall some required definition.

A continuous operation $(a,b) \mapsto a*b\colon [0,1]\times[0,1]\to[0,1]$ is called a t-norm if it is associative, commutative, monotone and 1 is a neutral element.

A function $M\colon X\times X\times(0,\infty)\to[0,1]$ is called a fuzzy metric on a set X, if it satisfies the conditions:

(i) $M(x,y,t)>0$;

(ii) $M(x,y,t)=1$ if and only if $x=y$;

(iii) $M(x,y,t)=M(y,x,t)$;

(iv) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$;

(v) the function $M(x,y,-):(0,\infty)\to[0,1]$ is continuous.

The value $M(x,y,t)$ can be interpreted as the probability that the distance between x and y does not exceed t.

A triple $(X,M,*)$ is called a fuzzy metric space [2]. If, instead of condition (iv) a fuzzy metric $M: X\times X\times(0,\infty)\to[0,1]$ satisfies a stronger condition

$$\text{(iv}'\text{)}\ M(x,y,t)*M(y,z,t)\le M(x,z,t),$$

then it is called a fuzzy ultrametric.

For every $x\in X$, every $r>0$ and $t>0$ let $B(x,r,t)=\{y\in X | M(x,y,t)>1-r\}$ (the ball of radius r centered at x for t).

It is known that the family of all balls is a base of topology on a fuzzy metric space.

By $P(X)$ we denote the set of probability measures of compact support on a fuzzy ultrametric space $(X,M,*)$. In [8], a fuzzy ultrametric $\hat{M}$ is defined on the set $P(X)$:

$$\hat{M}(\mu,\nu,t)=1-\inf\{r>0|\mu(B(x,r,t))=\nu(B(x,r,t),\forall x\in X\}$$

Let us reformulate this definition. Let $F_{r,t}=F_{r,t}(X)$ denote the set of functions on X which are constant on every ball $B(x,r,t)$.

Lemma 1. *For every $\mu,\nu\in P(X)$, we have*

$$\hat{M}(\mu,\nu,t)=1-\inf\left\{r>0|\int_X\varphi d\mu=\int_X\varphi d\nu,\forall\varphi\in F_{r,t}(X)\right\}.$$

The proof follows from the definition of the Lebesgue integral.

For every function $\varphi \in C(X)$, define the function $\bar{\varphi} : P(X) \to R$ by the formula $\bar{\varphi}(\mu) = \int_X \varphi d\mu$, for every measure $\mu \in P(X)$.

Lemma 2. *If $\varphi \in F_{r,t}(X)$, then $\bar{\varphi} \in F_{r,t}(P(X))$.*

Proof. Let $\hat{M}(\mu, \nu, t) < r$. Then the assertion of the lemma follows from the equality $\bar{\varphi}(\mu) = \int_X \varphi d\mu = \int_X \varphi d\nu = \bar{\varphi}(\nu)$.

Recall that the product of probability measures $\mu \in P(X)$ and $\nu \in P(Y)$ is the measure $\mu \otimes \nu \in P(X \times Y)$ that satisfies the condition: $(\mu \otimes \nu)(U \times V) = \mu(U)\nu(V)$ for every Borel sets $U \subset X$ and $V \subset Y$.

Let $n \in N$ and let G be a subgroup of the symmetric group S_n. Recall that the G-symmetric power of a space X is the quotient space $SP^n_G X$ of the product $X^n = X \times ... \times X$ with respect to the equivalence relation $\approx$:

$$(x_1, ..., x_n) \approx (x_{\sigma(1)}, ..., x_{\sigma(n)}), \quad \sigma \in G.$$

The equivalence class of the relation $\approx$ that contains $(x_1, ..., x_n)$, is denoted by $[x_1, ..., x_n]$. If $f : X \to Y$ is a map, then we define the map $SP^n_G f : SP^n_G X \to SP^n_G Y$ by the formula

$$SP^n_G f([x_1, ..., x_n]) = [f(x_1), ..., f(x_n)].$$

If $(X, M, *)$ is a fuzzy metric space, then one can define a fuzzy metric $\tilde{M}$ on the set $SP^n_G X$ by the formula:

$$\tilde{M}([x_1, ..., x_n], [y_1, ..., y_n], t) = \max_{\sigma \in G} \min_i M(x_i, y_{\sigma(i)}, t).$$

Proposition 1. *If $(X, M, *)$ is a fuzzy ultrametric space, then $(SP^n_G X, \tilde{M}, *)$ is also a fuzzy ultrametric space.*

Proof. We have to check only property (iv'). Let

$$[x_1, ..., x_n], [y_1, ..., y_n], [z_1, ..., z_n] \in SP^n_G X,$$

$t > 0$.

Then

$$\tilde{M}([x_1, ..., x_n], [y_1, ..., y_n], t) * \tilde{M}([y_1, ..., y_n], [z_1, ..., z_n], t) = \\ \max_{\sigma \in G} \min_i M(x_i, y_{\sigma(i)}, t) * \max_{\tau \in G} \min_j M(y_j, z_{\tau(j)}, t).$$

There exist $\bar{\sigma}, \bar{\tau} \in G$ such that

$$\max_{\sigma \in G} \min_i M(x_i, y_{\sigma(i)}, t) = \min_i M(x_i, y_{\bar{\sigma}(i)}, t),$$

$$\max_{\tau \in G} \min_j M(y_j, z_{\tau(j)}, t) = \min_j M(y_j, z_{\bar{\tau}(j)}, t).$$

Then, for every k we have

$$\begin{aligned}&\tilde{M}([x_1,...,x_n],[y_1,...,y_n],t) * \tilde{M}([y_1,...,y_n],[z_1,...,z_n],t) =\\ &= \min_i M(x_i, y_{\bar\sigma(i)},t) * \min_j M(y_j, z_{\bar\tau(j)},t) \le\\ &\le M(x_k, y_{\bar\sigma(k)},t) * M(y_{\bar\sigma(k)}, z_{\bar\tau(\bar\sigma(k))},t) \le M(x_k, z_{\bar\tau(\bar\sigma(k))},t),\end{aligned}$$

whence it follows that

$$\begin{aligned}&\tilde{M}([x_1,...,x_n],[y_1,...,y_n],t) * \tilde{M}([y_1,...,y_n],[z_1,...,z_n],t) \le\\ &\le \min_k M(x_k, z_{\bar\tau(\bar\sigma(k))},t) \le \max_{\rho\in G}\min_k M(x_k, z_{\rho(k)},t) =\\ &= \tilde{M}([x_1,...,x_n],[z_1,...,z_n],t).\end{aligned}$$

For each topological space X by $\exp X$ denote the set all nonempty compact subsets of the space X. In [6], for each fuzzy metric space $(X, M, *)$ a fuzzy metric Hausdorff on $\exp X$ is constructed. Let us recall the definition.

Let $x \in X$, $A \in \exp X$. Let $M(x, A, t) = \sup_{a\in A} M(x, a, t)$. Then for every $A, B \in \exp X$ we have

$$M_H(A, B, t) = \min\{\inf_{a\in A} M(a, B, t), \inf_{b\in B} M(b, A, t)\}.$$

Proposition 2. *If $(X, M, *)$ is a fuzzy metric space, then*

$$\begin{aligned}&M_H(A, B, t) = 1 - \inf\{r > 0 |\\ &A \cap B(x, r, t) \neq \emptyset \Leftrightarrow B \cap B(x, r, t) \neq \emptyset, \forall x \in X\}.\end{aligned} \tag{1}$$

Proof. Denote the right hand side of (1) by M_0. Let $M_0 < R$. Then for every $a \in A$ there exists $b \in B$ such that $b \in B(x, 1-R, t)$ and therefore $M(a, B, t) = \sup_{b\in B} M(a, b, t) \le 1 - (1 - R) = R$. In turn, $\inf_{a\in A}\sup_{b\in B} M(a, b, t) \le R$. Arguing similarly we conclude that $M_H(A, B, t) \le R$. Therefore, $M_H(A, B, t) \le M_0$.

The opposite inequality is established similarly.

Let $(X_i, M_i, *)$, $i = 1, 2$, be fuzzy metric spaces. A map $f : X_1 \to X_2$ is called nonexpanding if for every $x, y \in X_1$ and every $t > 0$ we have $M_2(f(x), f(y), t) \ge M_1(x, y, t)$. Fuzzy metric spaces and their nonexpanding maps form a category that we denote by $FMS(*)$. We denote by $UFMS(*)$ the subcategory of the category $FMS(*)$ whose objects are ultrametric spaces.

Proposition 3. *Let $(X_i, M_i, *)$, $i = 1, 2$, be fuzzy metric spaces and $f : X_1 \to X_2$ be a nonexpanding map. Then the map $SP^n_G f : SP^n_G X_1 \to SP^n_G X_2$ is also nonexpanding.*

Proof. See [7].

We therefore obtain that SP^n_G is a functor in the category $FMS(*)$. From Proposition 3 it also follows that SP^n_G is a functor in the category $UFMS(*)$.

Recall that the support of a probability measure $\mu \in P(X)$ is the minimal closed set $A \subset X$ satisfying the condition $\mu(X\backslash A) = 0$. The support of μ is denoted by $\mathrm{supp}(\mu)$.

Theorem 1. *Let $(X, M, *)$ be a fuzzy metric space. Then the map* $\mathrm{supp} : P(X) \to \exp X$ *is continuous.*

Proof. We use formula (1). Let $\hat{M}(\mu,\nu,t) > 1-R$. Then there exists $r < R$ for which $\mu(B(x,r,t)) = \nu(B(x,r,t), \forall x \in X$.

Let $B(y,r,t) \cap \mathrm{supp}(\mu) \neq \emptyset$, then $\mu(B(y,r,t)) \neq \emptyset$, and therefore $\nu(B(y,r,t)) \neq \emptyset$, whence $B(y,r,t) \cap \mathrm{supp}(\nu) \neq \emptyset$. Arguing similarly we obtain

$$B(y,r,t) \cap \mathrm{supp}(\mu) \neq \emptyset \Leftrightarrow B(y,r,t) \cap \mathrm{supp}(\nu) \neq \emptyset,$$

and therefore

$$M_H(\mathrm{supp}(\mu), \mathrm{supp}(\nu), t) > 1-r > 1-R,$$

whence the required inequality follows.

In other words, P is a functor with continuous supports in the category $UFMS(*)$. In fact, it is easy to see that supp is a natural transformation of the functor P into the functor exp. Now for every $M \in P2(X)$ define the map $\psi_X(M) : C(X) \to R$ by the formula: $\psi_X(M)(\varphi) = M(\bar{\varphi})$.

We are going to show that the support of the functional $\psi_X(M) : C(X) \to \mathbb{R}$ is compact. Recall that the support of a functional $f \colon C(X) \to \mathbb{R}$ is the minimal closed set $A \subset X$ such that $f(\varphi) = f(\psi)$, whenever $\varphi|A = \psi|A$. Directly from this definition it follows that the support of the functional $\psi_X(M)$ is equal to the set $A = \cup\{\mathrm{supp}(\mu)|\mu \in \mathrm{supp}(M)\}$. Theorem 1 and the fact that the union of any compact family of compact sets is again compact imply that the set A is compact.

Now it is easy to see that the functional $\psi_X(M) : C(X) \to \mathbb{R}$ is linear, is positive and regular, i.e., of norm 1, therefore is an element of $P(X)$.

Proposition 4. *The map $\psi_X : P^2(X) \to P(X)$ is nonexpanding. Here, the fuzzy metric $\hat{\hat{M}}$ is considered on the set $P^2(X)$.*

Proof. Let $N_1, N_2 \in P^2(X)$ and $\hat{\hat{M}}(N_1, N_2, t) > 1-R$, then for every $r > R$, every $t > 0$ and every $\varphi \in F_{r,t}(X)$, by Lemma 2 we have

$$\psi_X(N_1)(\varphi) = N_1(\bar{\varphi}) = N_2(\bar{\varphi}) = \psi_X(N_2)(\varphi).$$

This implies the inequality $\hat{M}(\psi_X(N_1), \psi_X(N_2), t) > 1-R$.

It is easy to show that $\psi = (\psi_X)$ is a natural transformation of the functor P^2 into the functor P in the category $UFMS(*)$.

3. Monads in the category of fuzzy ultrametric spaces

Recall some definitions from the category theory needed in the future. See, e.g., [1] for details. A monad in the category C is a triple $\mathbb{T} = (T, \eta, \mu)$ that consists of an endofunctor T in the category C and the natural transformations $\eta : 1_C \to T$, $\mu : T^2 \to T$ such that $\mu T(\eta) = \mu\eta_T = 1_T$ and $\mu T(\mu) = \mu\mu_T$.

The Kleisli category of a monad $\mathbb{T} = (T, \eta, \mu)$ in a category C is the category $C_{\mathbb{T}}$ defined by the conditions:

1. the class of objects of $C_{\mathbb{T}}$ coincides with that of C;
2. the set of morphisms from A to B in the category $C_{\mathbb{T}}$ coincides with the set of morphisms from A to $T(B)$ in the category C;
3. the composition $g * f$ of morphisms $f : A \to B$ and $g : B \to C$ in the category $C_{\mathbb{T}}$ is $\mu_C T(g) f$.

Denote by $I : C \to C_{\mathbb{T}}$ the functor which is constant on the objects and sends $f : A \to B$ to $\eta_B f$.

Let $F : C \to C$ be a functor. We say that a functor $\bar{F} : C_{\mathbb{T}} \to C_{\mathbb{T}}$ is an extension of a functor F onto the category $C_{\mathbb{T}}$, if $IF = \bar{F}I$.

We will need statement that makes continued existence criterion functors on the category Kleisli monad.

Proposition 5. *There is a one-to-one correspondence between the extensions of a functor $F : C \to C$ onto the Kleisli category $C_{\mathbb{T}}$ of the monad $\mathbb{T} = (T, \eta, \mu)$ in the category C and the natural transformations $\xi : FT \to TF$ satisfying the conditions:*
1) $\xi \circ F\eta = \eta_F$;
2) $\mu_F \circ \xi_T \circ T\xi = \xi \circ F\mu$.

Proof. See in [11].

Theorem 2. *The triple P$= (P, \delta, \psi)$ is a monad in the category $UFMS(*)$.*

Proof. Follows the schema of the proof of the corresponding result in [4].

The following result is a counterpart of one result from [4], which is proved herein for ultrametric spaces.

Theorem 3. *The G-symmetric power functor SP^n_G in the category $UFMS(*)$ has an extension onto the probability measure monad.*

Proof. For every fuzzy metric space X, denote the map $\xi_X : SP^n_G(P(X)) \to P(SP^n_G(X))$ by the formula:

$$\xi_X([\mu_1, ..., \mu_n]) = P(\pi_G)(\mu_1 \otimes ... \otimes \mu_n),$$

where $\pi_G : X^n \to SP^n_G(X)$ is the quotient map.

Let $[\mu_1, ..., \mu_n], [\nu_1, ..., \nu_n] \in SP^n_G P(X)$ and $\hat{M}([\mu_1, ..., \mu_n], [\nu_1, ..., \nu_n], t) > 1 - R$. Then there exists $\sigma \in G$ such that $\min_i \hat{M}(\mu_i, \nu_{\sigma(i)}, t) > 1 - R$, and therefore, for every $i = 1, ..., n$, we have $\hat{M}(\mu_i, \nu_{\sigma(i)}, t) > 1 - R$. Show that

$$\hat{M}(\mu_1 \otimes ... \otimes \mu_n, \nu_{\sigma(1)} \otimes ... \otimes \nu_{\sigma(n)}, t) > 1 - R.$$

Since $B'((x_1, ..., x_n), r, t) = \prod_{i=1}^{n} B(x_i, r, t)$ (here B' denotes the ball in the space X^n), the equalities $\mu_i(B(x_i, r, t)) = \nu_{\sigma(i)}(B(x_i, r, t))$, $i = 1, ..., n$, imply

$$(\mu_1 \otimes ... \otimes \mu_n)B'((x_1, ..., x_n), r, t) = \prod_{i=1}^{n} \mu_i(B(x_i, r, t)) = \prod_{i=1}^{n} \nu_{\sigma(i)}(B(x_i, r, t)) =$$
$$= (\nu_{\sigma(1)} \otimes ... \otimes \nu_{\sigma(n)})B'((x_1, ..., x_n), r, t),$$

and we obtain the required equality. Since the map π_G is nonexpanding, the map $P(\pi_G)$ is also nonexpanding. Therefore, the map ξ_X is nonexpanding, i.e. is a morphism of the category $UFMS(*)$.

The verification that the natural transformation $\xi = (\xi_X)$ satisfies the properties of Proposition 5 follows the arguments of [12].

Theorem 4. *Let $H \subset G$ be subgroups of the symmetric group S_n. A natural transformation $\zeta_{GH} : SP^n_G \to SP^n_H$ in the category $UFMS(*)$ is also a natural transformation of the extended functors SP^n_G, SP^n_H in the category $UFMS(*)_{\mathbb{P}}$.*

Proof. The proof follows that of Proposition 5.6.9 from [10].

Finally, we formulate an open problem of finding counterparts of the results of this paper for fuzzy metric spaces in the sense of [5]. This notion is slightly less restrictive as in condition (v) it is required only that the function $M(x, y, -) : (0, \infty) \to [0, 1]$ be left-continuous.

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