Topology and its Applications 157 (2010) 724–729

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Metrization of free groups on ultrametric spaces

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ARTICLE INFO

Article history:
Received 7 December 2008
Received in revised form 30 July 2009

MSC:
54E35
46S10
54H11
18C15

Keywords:
Free topological group
Invariant metric
Ultrametric
Monad
Category of ultrametric spaces

ABSTRACT

We consider ultrametrizations of free topological groups of ultrametric spaces. A construction is defined that determines a functor in the category UMET_1 of ultrametric spaces of diameter $\leqslant 1$ and nonexpanding maps. This functor is the functorial part of a monad in UMET_1 and we provide a characterization of the category of its algebras.

1. Introduction

The locally invariant topologies on the free groups, i.e. the topologies generated by the invariant pseudometrics is an object of study in many papers (see, e.g., [8]).

Recall that a metric d on a set X is said to be an *ultrametric* (a non-Archimedean metric, in another terminology) if the following strong triangle inequality holds:

$$d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$$

for all $x, y, z \in X$.

In [11], an ultrametric is defined on the set of probability measures with compact supports defined on an ultrametric space. In [4], some properties of the obtained functor $\mathcal{M}eas$ in the category of (complete) ultrametric spaces of diameter $\leqslant 1$ are established. It is pointed out in [4] that the functor $\mathcal{M}eas$ fits naturally in the metric approach to the programming language semantics.

Similar construction is defined for the so called idempotent measures in [5,12]. The aim of the present paper is to define a natural ultrametric on the free topological group in the sense of Markov [7] of an ultrametric space.

Recall that a (pseudo)metric d on a group G is called *left invariant* (respectively *right invariant*) if $d(x, y) = d(gx, gy)$ (respectively $d(x, y) = d(xg, yg)$), for all $x, y, g \in G$. A metric d is called *invariant* if d is both left and right invariant.

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.08.015

It is a classical construction due to Graev [3] that extends every (pseudo)metric on a Tychonov space X to an invariant (pseudo)metric on the free topological group $F(X)$. Another construction, which also works for a wide class of topological algebras, is proposed by Świerczkowski [10].

We prove that the obtained ultrametrization of the free groups over ultrametric spaces determines a functor in the category of ultrametric spaces and nonexpanding maps. Moreover, this functor is the functorial part of a monad in the category of ultrametric spaces of diameter $\leqslant 1$ and nonexpanding maps. One of the results of the paper provides a characterization of the category of algebras of this monad. This is precisely the category of groups endowed with invariant ultrametric of diameter $\leqslant 1$ and nonexpanding homomorphisms.

2. Preliminaries

Let X be a Tychonov space. Recall that a free topological group (in the sense of Markov) of X is a topological group, denoted $F(X)$, satisfying the following properties:

(1) X is a subspace of $F(X)$;
(2) any continuous map of X into a topological group G admits a unique extension which is a continuous homomorphism of $F(X)$ into G.

Replacing, in the above definition, the term ‘group’ by ‘abelian group’, we obtain the definition of the free abelian topological group (usually denoted by $A(X)$).

It is well known that the free topological groups exist. The construction of free topological group is functorial. Given a map $f : X \to Y$ of Tychonov spaces, we define the homomorphism $F(f) : F(X) \to F(Y)$ as the unique extension of $f : X \to Y \hookrightarrow F(Y)$.

The Graev metric on $F(X)$ is the maximal invariant pseudometric that induces the initial metric on X.

It is also known that algebraically $F(X)$ is a free group over X. Every $u \in F(X)$ can be represented in the form $u = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$, where $x_1, \ldots, x_n \in X$, $\varepsilon_i \in \{-1, 1\}$ and $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ is an irreducible word in the sense that, if $x_i = x_{i+1}$, then $\varepsilon_i = \varepsilon_{i+1}$. The set $\{x_1, \ldots, x_n\}$ is then called the *support* of u and is denoted $\mathrm{supp}(u)$.

Let

$$F_0(X) = \left\{ u = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} \in F(X) \;\middle|\; \sum_{i=1}^{n} \varepsilon_i = 0 \right\}.$$

It is well known that $F_0(X)$ is a normal subgroup of $F(X)$; actually, $F_0(X)$ is the kernel of the homomorphism $\alpha_X : F(X) \to \mathbb{Z}$ which extends the constant map $X \to \{1\} \subset \mathbb{Z}$. Note that, for every map $f : X \to Y$ of Tychonov spaces, the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(X) \\ & {\scriptstyle \alpha_X}\searrow \quad \swarrow{\scriptstyle \alpha_Y} & \\ & \mathbb{Z} & \end{array}$$

is commutative.

Let UMET_1 denote the category whose objects are ultrametric spaces of diameter $\leqslant 1$ and whose morphisms are nonexpanding maps.

We recall some definitions concerning monads; see, e.g., [1] for details. Let T be an endofunctor in a category $\mathcal{C}$. By T^n we denote the n-th iteration of T, $T^n(X) = T(T(\ldots T(X)\ldots))$ (n times). If $\eta : 1_{\mathcal{C}} \to T$ and $\mu : T^2 \to T$ are natural transformations, then $\mathbb{T} = (T, \eta, \mu)$ is called a *monad* if and only if the diagrams

$$\begin{array}{ccc} T & \xrightarrow{\eta T} & T^2 \\ {\scriptstyle T\eta}\downarrow & \searrow^{1_T} & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array} \qquad \begin{array}{ccc} T^3 & \xrightarrow{\mu T} & T^2 \\ {\scriptstyle T\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array}$$

are commutative. Then η is called the *unity* and μ the *multiplication* of $\mathbb{T}$. The functor T is often referred to as the *functorial part* of $\mathbb{T}$.

For an arbitrary monad $\mathbb{T} = (T, \eta, \mu)$ in $\mathcal{C}$ a pair (X, ξ), where $\xi : TX \to X$ is a morphism in $\mathcal{C}$, is called a $\mathbb{T}$-*algebra* if and only if the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\eta X} & TX \\ & \searrow^{1_X} & \downarrow \xi \\ & & X \end{array} \qquad \begin{array}{ccc} T^2X & \xrightarrow{\mu X} & TX \\ T\xi \downarrow & & \downarrow \xi \\ TX & \xrightarrow{\xi} & X \end{array}$$

commute.

The morphism $\xi : TX \to X$ is then referred to as the *structure morphism* of the $\mathbb{T}$-algebra (X, ξ).

A morphism $f : X \to X'$ in $\mathcal{C}$ is said to be a *morphism* of $\mathbb{T}$-algebras $(X, \xi) \to (X', \xi')$ if the diagram

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TX' \\ \xi \downarrow & & \downarrow \xi' \\ X & \xrightarrow[f]{} & X' \end{array} \tag{1}$$

is commutative.

It is easy to see that $\mathbb{T}$-algebras and their morphisms form a category. This category is denoted by $\mathcal{C}^{\mathbb{T}}$.

3. Main result

Let (X, d) be a metric space. If $A \subset X$ and $r > 0$, then the r-neighborhood of A is the set $O_r(A) = \{x \in X \mid d(x, a) < r$ for some $a \in A\}$. We write $O_r(x)$ if $A = \{x\}$.

Let (X, d) be an ultrametric space. Given $r > 0$, we denote by $\mathcal{F}_r = \mathcal{F}_{X,r}$ the decomposition of X into the disjoint family of balls of radius r. We denote by $q_r = q_{X,r} : X \to X/\mathcal{F}_r$ the quotient map. One can regard $X/\mathcal{F}_r$ as a discrete topological space. We will use the following fact which easily follows from elementary properties of ultrametrics: if $r < r'$, then the map $q_{r'}$ can be factored through q_r.

We now suppose that $\operatorname{diam}(X) \leqslant 1$. We define the function $\hat{d} : F(X) \times F(X) \to \mathbb{R}$ as follows:

$$\hat{d}(u, v) = \begin{cases} 1, & \text{if}\, \alpha(u) \neq \alpha(v), \\ \inf\{r > 0 \mid F(q_r)(u) = F(q_r)(v)\}, & \text{if}\, \alpha(u) = \alpha(v). \end{cases}$$

Theorem 3.1. *The function $\hat{d}$ is an invariant continuous ultrametric on the topological group $F(X)$.*

Proof. We first note that $\hat{d}$ is well defined. Let $u, v \in F(X)$. If $\alpha(u) \neq \alpha(v)$, then there is nothing to prove. Suppose that $\alpha(u) = \alpha(v)$ and let $R > \operatorname{diam}(\operatorname{supp}(u) \cup \operatorname{supp}(v))$. Then there is $x \in X$ such that $O_R(x) \supset \operatorname{supp}(u) \cup \operatorname{supp}(v)$ and we easily see that $F(q_R)(u) = F(q_R)(v) = (q_R(x))^{\alpha(u)}$. Note that $0 \leqslant \hat{d}(u, v) \leqslant 1$.

Suppose that $u \neq v$. If $\alpha(u) \neq \alpha(v)$, then $\hat{d}(u, v) = 1$. If $\alpha(u) = \alpha(v)$, then $a = \min\{d(x, y) \mid x, y \in \operatorname{supp}(u) \cup \operatorname{supp}(v)$, $x \neq y\} > 0$. If $0 < r < a$, then the restriction of the map q_r to $\operatorname{supp}(u) \cup \operatorname{supp}(v)$ is injective, whence $\hat{d}(u, v) > 0$.

The symmetry of the function $\hat{d}$ immediately follows from the definition.

Let us verify the strong triangle inequality for $\hat{d}$. Let $u, v, w \in F(X)$, $\hat{d}(u, v) = a$, $\hat{d}(v, w) = b$. If $\max\{a, b\} = 1$, then the inequality is obviously satisfied. Thus, we may suppose that $\max\{a, b\} < 1$ and therefore $\alpha(u) = \alpha(v) = \alpha(w)$. Then, for every $r > \max\{a, b\}$, we have $F(q_r)(u) = F(q_r)(v) = F(q_r)(w)$, whence $\hat{d}(u, w) \leqslant r$ and we are done.

In order to prove that the obtained ultrametric is left invariant it suffices to show that $\hat{d}(uv, uw) \leqslant \hat{d}(v, w)$, for every $u, v, w \in F(X)$. If $\hat{d}(v, w) = 1$, there is nothing to prove. Otherwise, there is $r \in (0, 1)$ such that $\hat{d}(v, w) < r$ and therefore $F(q_r)(v) = F(q_r)(w)$. Then also

$$F(q_r)(uv) = F(q_r)(u)F(q_r)(v) = F(q_r)(u)F(q_r)(w) = F(q_r)(uw),$$

which implies $\hat{d}(uv, uw) < r$ and we obtain the required inequality.

One can similarly prove that $\hat{d}$ is right invariant and this finishes the proof. □

In the sequel, we tacitly assume that the free group $F(X)$ of an ultrametric space (X, d) is endowed with the metric $\hat{d}$.

Let $f : X \to Y$ be a morphism in the category UMET_1. The induced map $F(f) : F(X) \to F(Y)$ is defined as follows. If $u \in F(X)$ and $u = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$, where $x_1, \dots, x_n \in X$, $\varepsilon_i \in \{-1, 1\}$, then we let $F(f)(u) = f(x_1)^{\varepsilon_1} \dots f(x_n)^{\varepsilon_n} \in F(Y)$.

Proposition 3.2. *If $f : (X, d) \to (Y, \varrho)$ is a morphism in* UMET_1 *then the induced homomorphism is also a morphism in* UMET_1.

Proof. Suppose that $u, v \in F(X)$ and $\hat{d}(u, v) \leqslant c$. If $\hat{d}(u, v) = 1$, then there is nothing to prove, therefore we assume that $c < 1$. Then $\alpha(F(f)(u)) = \alpha(F(f)(v))$ and, for any $c' > c$, since the map f is nonexpanding, there exists $g : X/\mathcal{F}_{c'} \to Y/\mathcal{F}_{c'}$ such that $q_{Y,c'} f = g q_{X,c'}$, whence

$$F(q_{Y,c'})F(f)(u) = F(q_{Y,c'}f)(u) = F(gq_{X,c'})(u) = F(g)F(q_{X,c'})(u) = F(g)F(q_{X,c'})(v) = F(q_{Y,c'})F(f)(v)$$

and we see that $\hat{d}(F(f)(u), F(f)(v)) < c'$. We finally conclude that $\hat{d}(F(f)(u), F(f)(v)) \leqslant c$ and therefore $F(f)$ is nonexpanding. □

We therefore obtain a functor F in the category UMET_1.

Lemma 3.3. *Let G be a topological group whose topology is generated by an invariant ultrametric d. If $a_1, \dots, a_n, b_1, \dots, b_n \in G$, then*

$$d(a_1 \dots a_n, b_1 \dots b_n) \leqslant \max\{d(a_i, b_i) \mid i = 1, \dots, n\}.$$

Proof. This follows from the invariance, strong triangle inequality and induction by n:

$$\begin{aligned} d(a_1 \dots a_n, b_1 \dots b_n) &\leqslant \max\{d(a_1 \dots a_n, a_1 \dots a_{n-1}b_n), d(a_1 \dots a_{n-1}b_n, b_1 \dots b_n)\} \\ &\leqslant \max\{d(a_n, b_n), d(a_1 \dots a_{n-1}, b_1 \dots b_{n-1})\} \\ &\leqslant \max\{d(a_i, b_i) \mid i = 1, \dots, n\}. \qquad \square \end{aligned}$$

Proposition 3.4. *Let G be a topological group whose topology is generated by an invariant ultrametric d and $\operatorname{diam} G \leqslant 1$. Let $\xi : F(G) \to G$ denote the unique homomorphism that extends the identity map of G. Then ξ is nonexpanding.*

Proof. For the sake of convenience, denote by $*$ the operation in G. Let $u, v \in F(X)$, $\hat{d}(u, v) \leqslant c$. Then, for every $c' > c$, we have $F(q_{c'})(u) = F(q_{c'})(v)$. Without loss of generality, one may assume that the map $q_{c'}$ is a retraction onto a subset of X (thus $q_{c'}(a) \in A$, for any $a \in A \in \mathcal{F}_{c'}$). If $u = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$, then $F(q_{c'})(u) = q_{c'}(x_1)^{\varepsilon_1} \dots q_{c'}(x_n)^{\varepsilon_n}$. By Lemma 3.3, since $d(x_i, q_{c'}(x_i)) < c'$, for every $i = 1, \dots, n$, we see that

$$d\bigl(x_1^{\varepsilon_1} * \dots * x_n^{\varepsilon_n}, q_{c'}(x_1)^{\varepsilon_1} * \dots * q_{c'}(x_n)^{\varepsilon_n}\bigr) < c'.$$

Similarly, if $v = y_1^{\eta_1} \dots y_m^{\eta_m}$, then

$$d\bigl(y_1^{\eta_1} * \dots * y_m^{\eta_m}, q_{c'}(y_1)^{\eta_1} * \dots * q_{c'}(y_m)^{\eta_m}\bigr) < c'.$$

Since $q_{c'}(x_1)^{\varepsilon_1} \dots q_{c'}(x_n)^{\varepsilon_n} = q_{c'}(y_1)^{\eta_1} \dots q_{c'}(y_m)^{\eta_m}$, we see that

$$q_{c'}(x_1)^{\varepsilon_1} * \dots * q_{c'}(x_n)^{\varepsilon_n} = q_{c'}(y_1)^{\eta_1} * \dots * q_{c'}(y_m)^{\eta_m}$$

and therefore

$$\begin{aligned} d\bigl(\xi(u), \xi(v)\bigr) &= d\bigl(x_1^{\varepsilon_1} * \dots * x_n^{\varepsilon_n}, y_1^{\eta_1} * \dots * y_m^{\eta_m}\bigr) \\ &\leqslant \max\bigl\{d\bigl(x_1^{\varepsilon_1} * \dots * x_n^{\varepsilon_n}, q_{c'}(x_1)^{\varepsilon_1} * \dots * q_{c'}(x_n)^{\varepsilon_n}\bigr), d\bigl(q_{c'}(x_1)^{\varepsilon_1} * \dots * q_{c'}(x_n)^{\varepsilon_n}, y_1^{\eta_1} * \dots * y_m^{\eta_m}\bigr)\bigr\} \\ &< c'. \end{aligned}$$

This implies that $d(\xi(u), \xi(v)) \leqslant c$ and the map ξ is nonexpanding. □

We apply this statement in the situation, when $G = F(X)$, for an ultrametric space (X, d) of diameter $\leqslant 1$. Then the metric $\hat{d}$ on the group $F(X)$ generates the metric $\hat{\hat{d}}$ on $F(F(X)) = F^2(X)$.

Corollary 3.5. *The natural map $\mu_X : (F^2(X), \hat{\hat{d}}) \to (F(X), \hat{d})$ is nonexpanding.*

Given an ultrametric space (X, d) of diameter $\leqslant 1$, we denote by $\eta_X : X \to F(X)$ the inclusion map. Note that η_X is nonexpanding. Indeed, given $x, y \in X \subset F(X)$ with $d(x, y) < r$, note that $\alpha(\eta_X(x)) = \alpha(\eta_X(y))$ and the inequality $\hat{d}(\eta_X(x), \eta_X(y)) < r$ follows from the fact that $q_r(x) = q_r(y)$.

One can easily see that $\eta = (\eta_X)$ is a natural transformation of the identity functor in the category UMET_1 into the functor F.

Theorem 3.6. *The triple $\mathbb{F} = (F, \eta, \mu)$ is a monad on the category UMET_1.*

Proof. This is a consequence of the above results as well as well-known algebraic facts concerning the free group functor (see, e.g., [6]). □

Proposition 3.7. *Let G be a topological group whose topology is generated by an invariant ultrametric and* $\operatorname{diam} G \leqslant 1$. *Then (G, ξ), where $\xi : F(G) \to G$ is the natural map, is an $\mathbb{F}$-algebra. If G' is also topological group whose topology is generated by an invariant ultrametric,* $\operatorname{diam} G \leqslant 1$, *and $h : G \to G'$ is a nonexpanding homomorphism, then h is a morphism of the $\mathbb{F}$-algebra (G, ξ) into the $\mathbb{F}$-algebra (G', ξ'), where $\xi' : F(G') \to G'$ is the natural map.*

Proof. That (G, ξ) is an $\mathbb{F}$-algebra is a consequence of Proposition 3.4 and simple calculations. Also, Proposition 3.2 and direct verification of the commutativity of the diagram that corresponds to diagram (1) from the definition of morphism of algebras finishes the proof. □

Proposition 3.8. *Let (G, d, ξ) be an $\mathbb{F}$-algebra. Then there exists a unique group structure on G such that d is an invariant ultrametric on G and the map $\xi : F(G) \to G$ is a homomorphism. If (G', ξ') is also an $\mathbb{F}$-algebra and $f : G \to G'$ is a morphism of $\mathbb{F}$-algebras, then f is a homomorphism of the mentioned group structures on G and G'.*

Proof. We define the binary operation $*$ on G as follows: $g * h = \xi(gh)$. A routine verification (see, e.g., [6]) shows that $*$ is a group operation on G.

Note that, since both $\xi : F(G) \to G$ and $\eta_G : G \to F(G)$ are morphisms in UMET_1 and therefore nonexpanding maps, we obtain that the map η_G (i.e. the natural inclusion of G into $F(G)$) is an isometric embedding.

Let $x, g, h \in G$, then $\hat{d}(xg, xh) = \hat{d}(g, h)$ and therefore, since the homomorphism ξ is nonexpanding, we see that

$$d(x * g, x * h) = d\big(\xi(xg), \xi(xh)\big) \leqslant \hat{d}(xg, xh) = \hat{d}(g, h) = d(g, h).$$

This demonstrates that the ultrametric d is left invariant.

The rest of the proof is left to the reader. □

Summing up the above results, we obtain the following result that contains a description of the category $\mathbb{F}$-algebras.

Theorem 3.9. *The category* $\mathrm{UMET}_1^{\mathbb{F}}$ *is isomorphic to the category whose objects are groups endowed with the invariant ultrametric of diameter* $\leqslant 1$ *and whose morphisms are nonexpanding homomorphisms.*

Remark 3.10. The above results have also their counterparts for the case of the functor of free abelian group. The proofs can be obtained by mutatis mutandis.

4. Remarks and open problems

The results of the present paper can be generalized in different directions.

Let X be a Tychonov space and $e \in X$. Recall that a free topological group (in the sense of Graev) of X is a topological group with e as the unit element, denoted $F(X, e)$, satisfying the following properties:

(1) X is a subspace of $F(X, e)$;
(2) any continuous map of X into a topological group G that sends e to the unit of G admits a unique extension which is a continuous homomorphism of $F(X, e)$ into G.

Question 4.1. Is there a counterpart of the above construction for the Graev free topological groups of ultrametric spaces?

A natural question arises of extension of the obtained results over the case of free paratopological groups. Recall that paratopological groups are groups endowed with a topology that makes the multiplication (but not necessarily the inversion) continuous.

The existence of the free paratopological groups was proved in [9] by using a Graev type extension of quasi(pseudo)-metrics from a topological space onto the free group of its underlying set. Recall that a function $d : X \times X \to \mathbb{R}_+$ is a quasipseudometric on X if satisfies: (1) $d(x, x) = 0$ and (2) $d(x, y) \leqslant d(x, z) + d(z, y)$ (the triangle inequality; note that the order of arguments is essential). A natural asymmetric counterpart of the notion of ultrapseudometric is that of pseudo-quasi ultrametric which is obtained when we replace (2) with the property (2') $d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$ (the strong triangle inequality; again, the order of arguments is essential). This is called a generalized ultrametric in [2].

Question 4.2. Is there a counterpart of the construction described in this paper for the free paratopological groups?

Recall that a uniform structure on a set is called *non-Archimedean*, if possesses a base consisting of binary relation. Every such structure can be generated by a family of ultra(pseudo)metrics therefore the construction of the present paper can be applied to free groups of spaces endowed with non-Archimedean uniform structure.

Recall the Świerczkowski construction for the invariant metric on the free groups over a metric space (X,d). Given $u,v\in F(X)$, one can find a word $w(\alpha_1,\dots,\alpha_n)$ in the letters $\alpha_1,\dots,\alpha_n$ (each α_i can occur many times in the expression for w) and $a_1,\dots,a_n,b_1,\dots,b_n\in X$ such that $w(a_1,\dots,a_n)=u$, $w(b_1,\dots,b_n)=v$. We define the Świerczkowski metric, $\tilde{d}$, on $F(X)$ by letting

$$\tilde{d}(u,v)=\inf\left\{\sum_{i=1}^{n} d(a_i,b_i)\right\}, \tag{2}$$

where infimum is taken over all the words w and all the choices (a_i), (b_i).

Suppose that (X,d) is an ultrametric space. We conjecture that, replacing, in the above definition, (2) by

$$\tilde{d}(u,v)=\inf\bigl\{\max\bigl\{d(a_1,b_1),\dots,d(a_n,b_n)\bigr\}\bigr\}, \tag{3}$$

we also obtain an invariant ultrametric on the set $F(X)$ that extends the original metric on X.

Note that the Świerczkowski construction can be applied also to a wide class of topological algebras and we expect that its modification that uses a suitable version of (3) will provide ultrametrization of free topological algebras over ultrametric spaces.

Acknowledgements

The authors express their gratitude to the referee for careful reading of the manuscript and correcting mistakes in it.

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